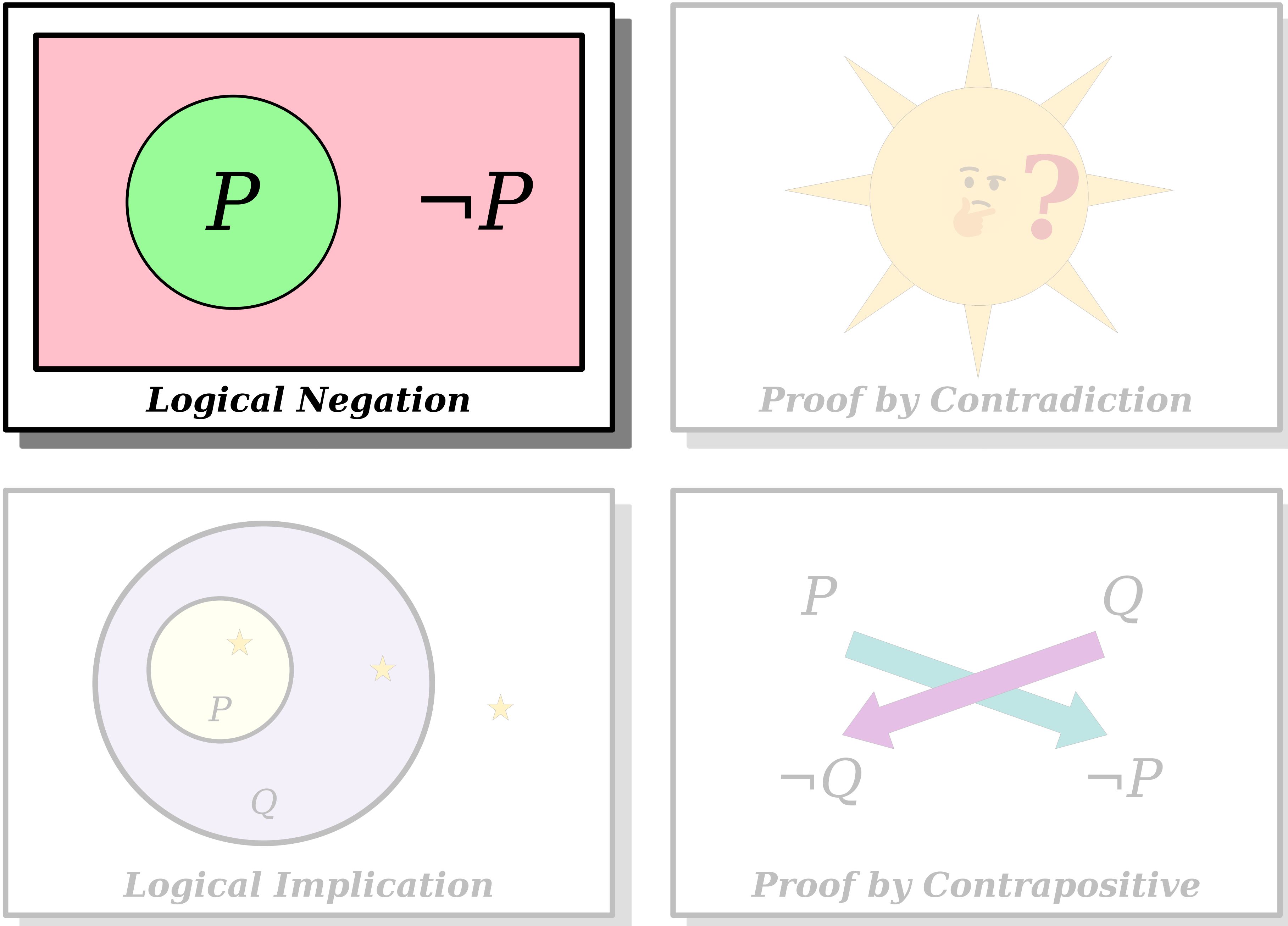
CS103
Winter 2026

Lecture 02:

Indirect Proofs

Indirect Proofs: A Story in Four Acts



Act I

Logical Negation

Propositions and Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
    - If n is an even integer, then n^2 is an even integer.
    - $\emptyset = \mathbb{R}$.
- The ***negation*** of a proposition X is a proposition that is true when X is false and is false when X is true.
- For example, consider the proposition “it is snowing outside.”
    - Its negation is “it is not snowing outside.”
    - Its negation is *not* “it is sunny outside.” ⚠️
    - Its negation is *not* “we’re in the Bay Area.” ⚠️

Propositions and Negations

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- Some examples:
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  - $\emptyset = \mathbb{R}$.
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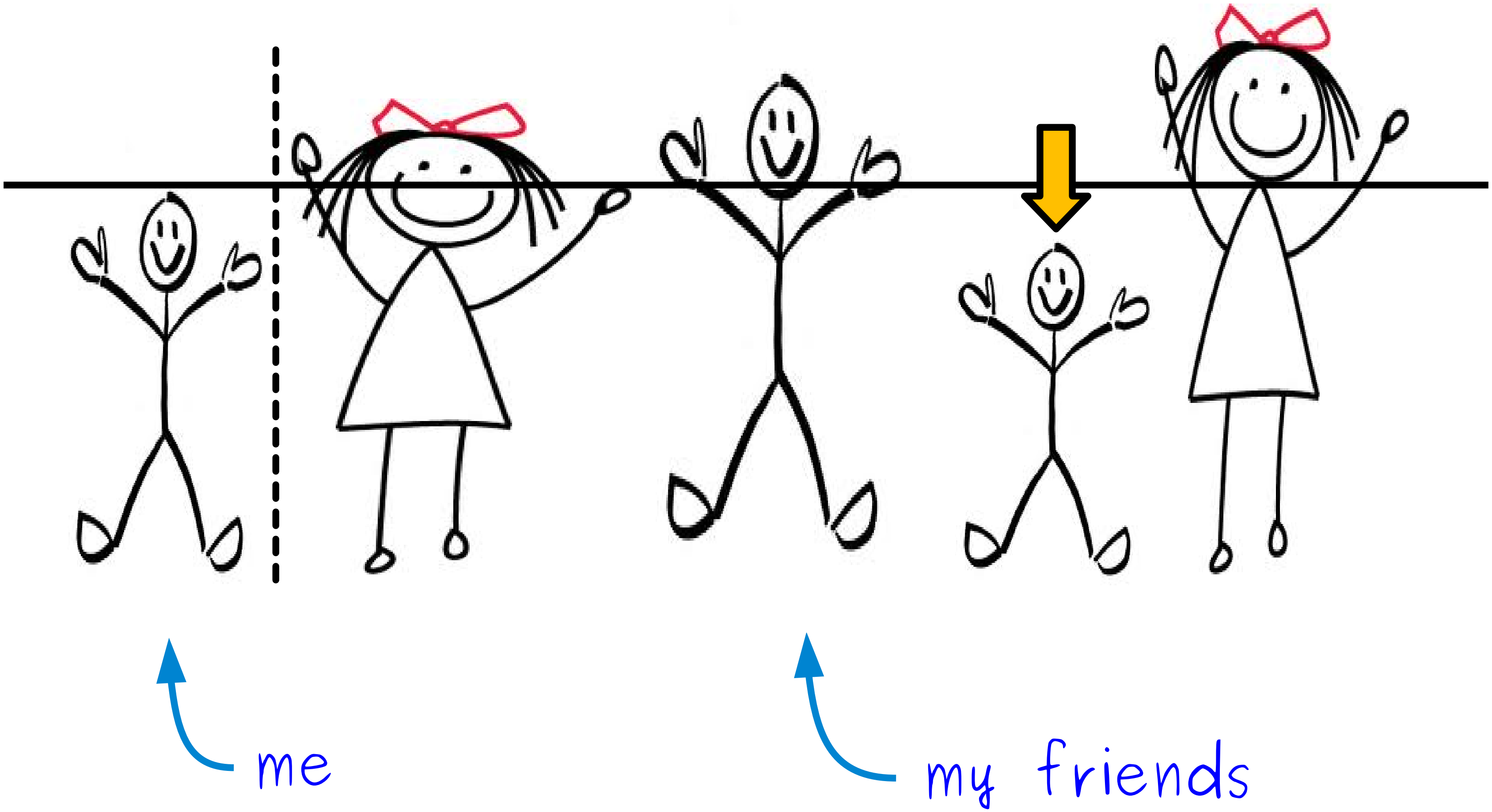
So, how do we find the negation of a proposition?

- For example, consider the proposition "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside." ⚠️
  - Its negation is *not* "we're in the Bay Area." ⚠️

Negating Statements

"All my friends are taller than me."



Negating Statements

Every *P* is a *Q*.

The negation of a ***universal*** statement

is an ***existential*** statement.

There is a *P* that is not a *Q*.

Negating Statements

For all *x*, *P*(*x*) is true.

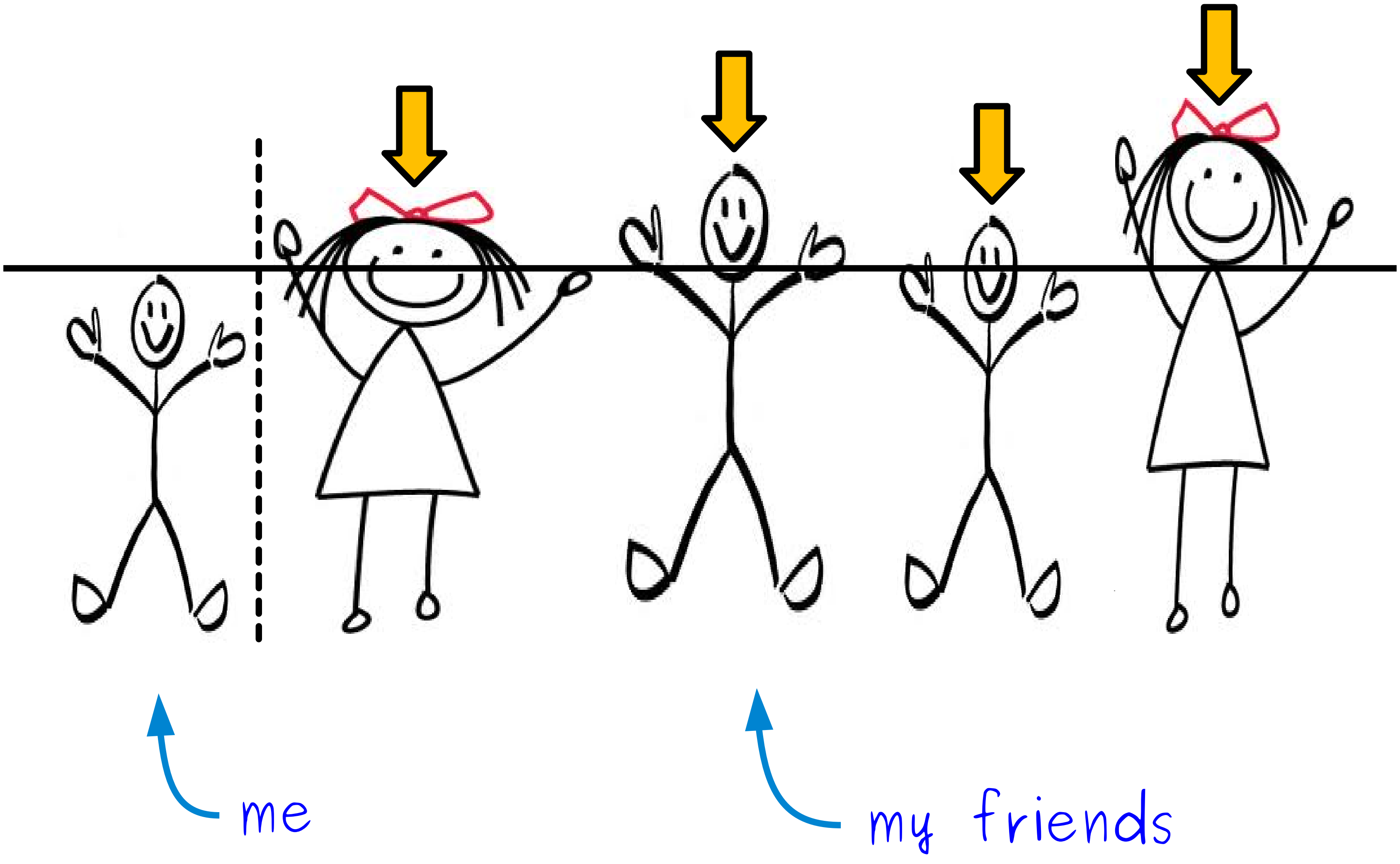
The negation of a ***universal*** statement

is an ***existential*** statement.

There exists an *x* where *P*(*x*) is false.

Negating Statements

"Some friend is shorter than me."



Negating Statements

There exists a *P* that is a *Q*.

The negation of an ***existential*** statement

is a ***universal*** statement.

Every *P* is not a *Q*.

Negating Statements

There exists an *x where*
***P*(*x*) is true.**

The negation of an ***existential*** statement

is a ***universal*** statement.

For all *x*,
***P*(*x*) is false.**

Negating Statements

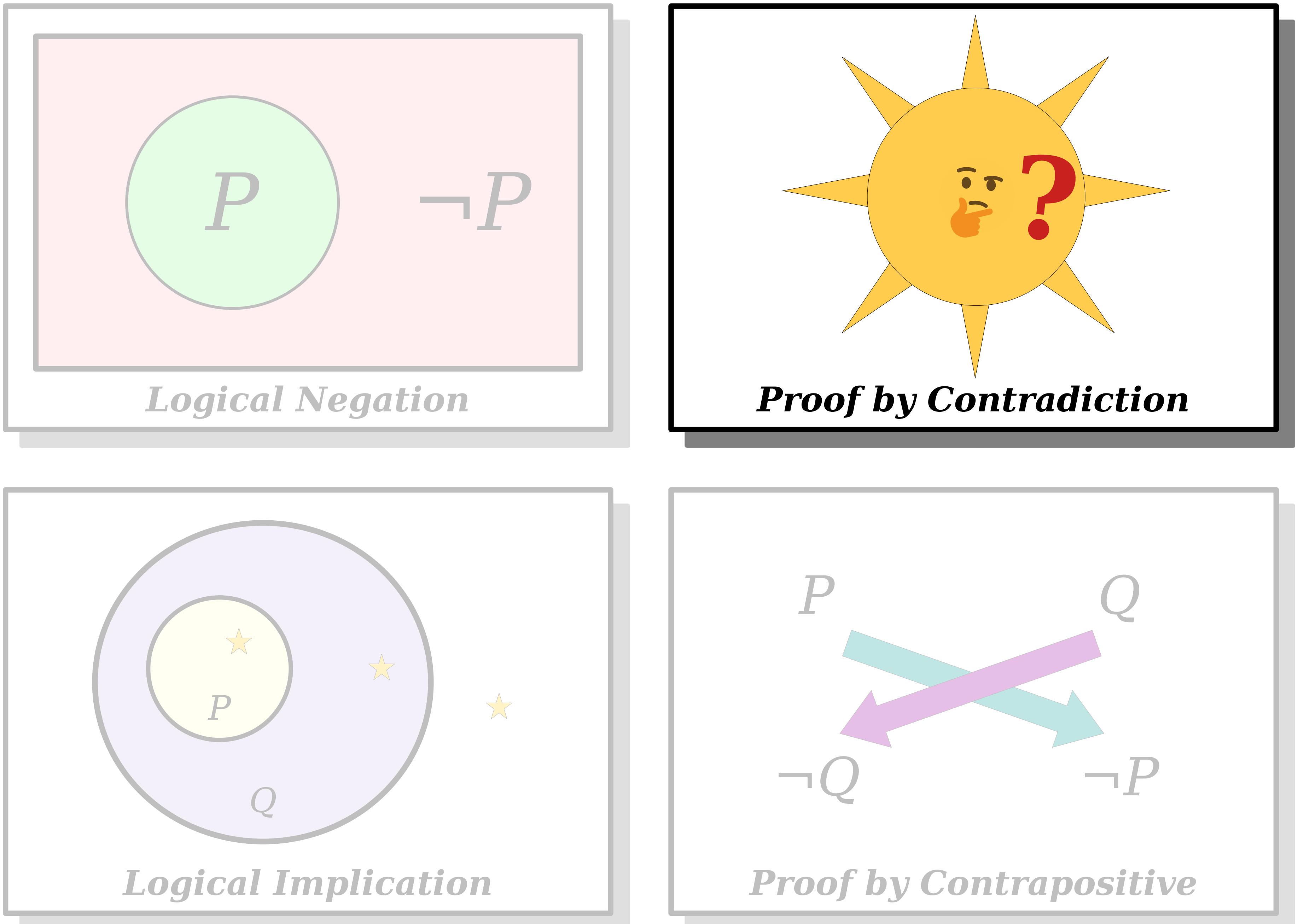
- Your turn! What's the negation of the following statement?

"Every brown dog loves every orange cat."

- Answer:

"There is a brown dog that doesn't love some orange cat."

Indirect Proofs: A Story in Four Acts



Act II

Proof by Contradiction

First, let’s reflect on the **direct proof** technique we saw Wednesday.

Our First Proof! (from Wednesday)

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

start by asking the reader to assume that ***P*** is true.

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Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

If we apply **sound logic** (using definitions, algebra, etc.) all the statements that follow are **also true.**

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Proof: Assume … that n^2 is even.

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More generally speaking,
the process looks like this:

Direct Proof

We start with a statement (or statements) we know (or assume) to be true.

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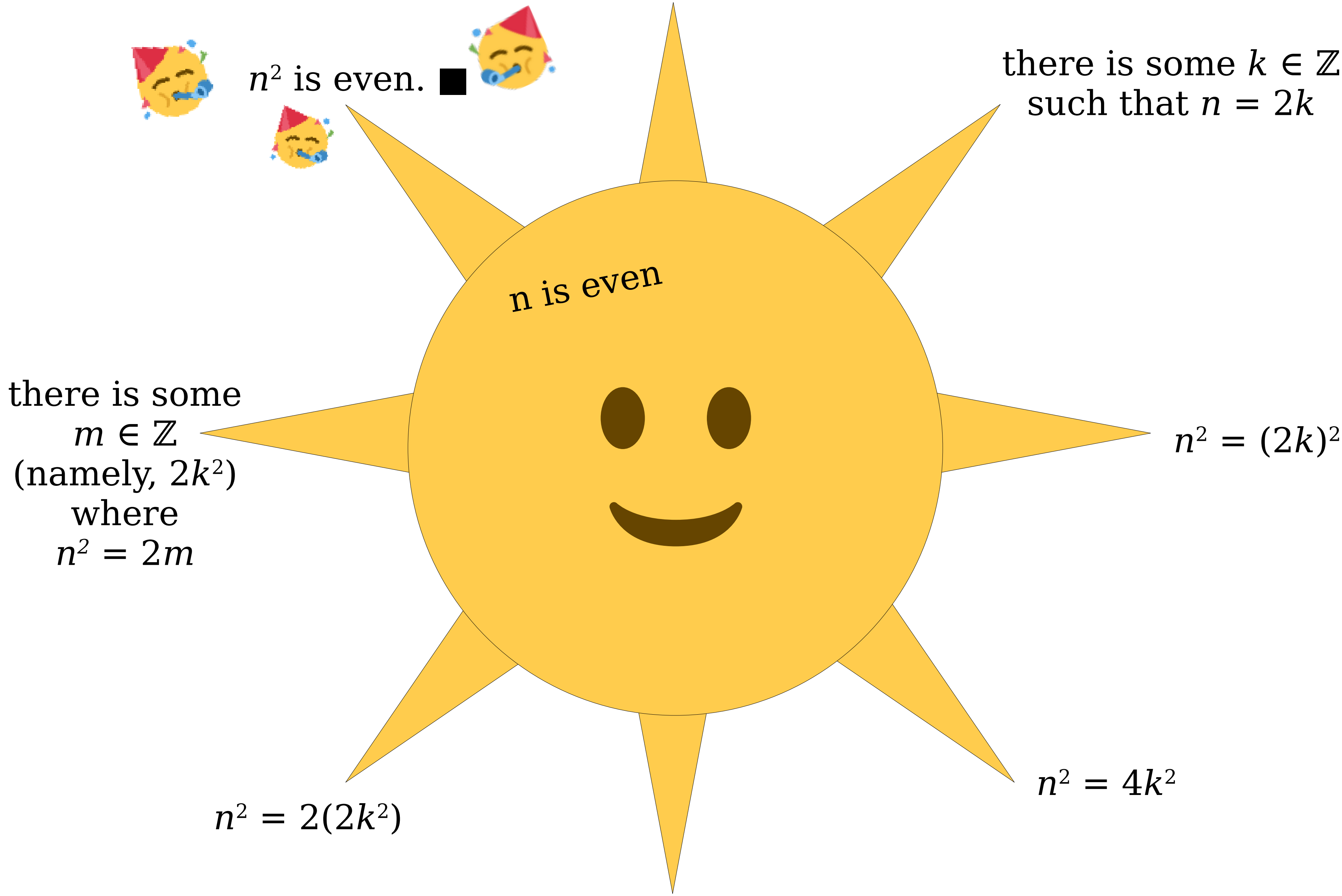
Direct Proof

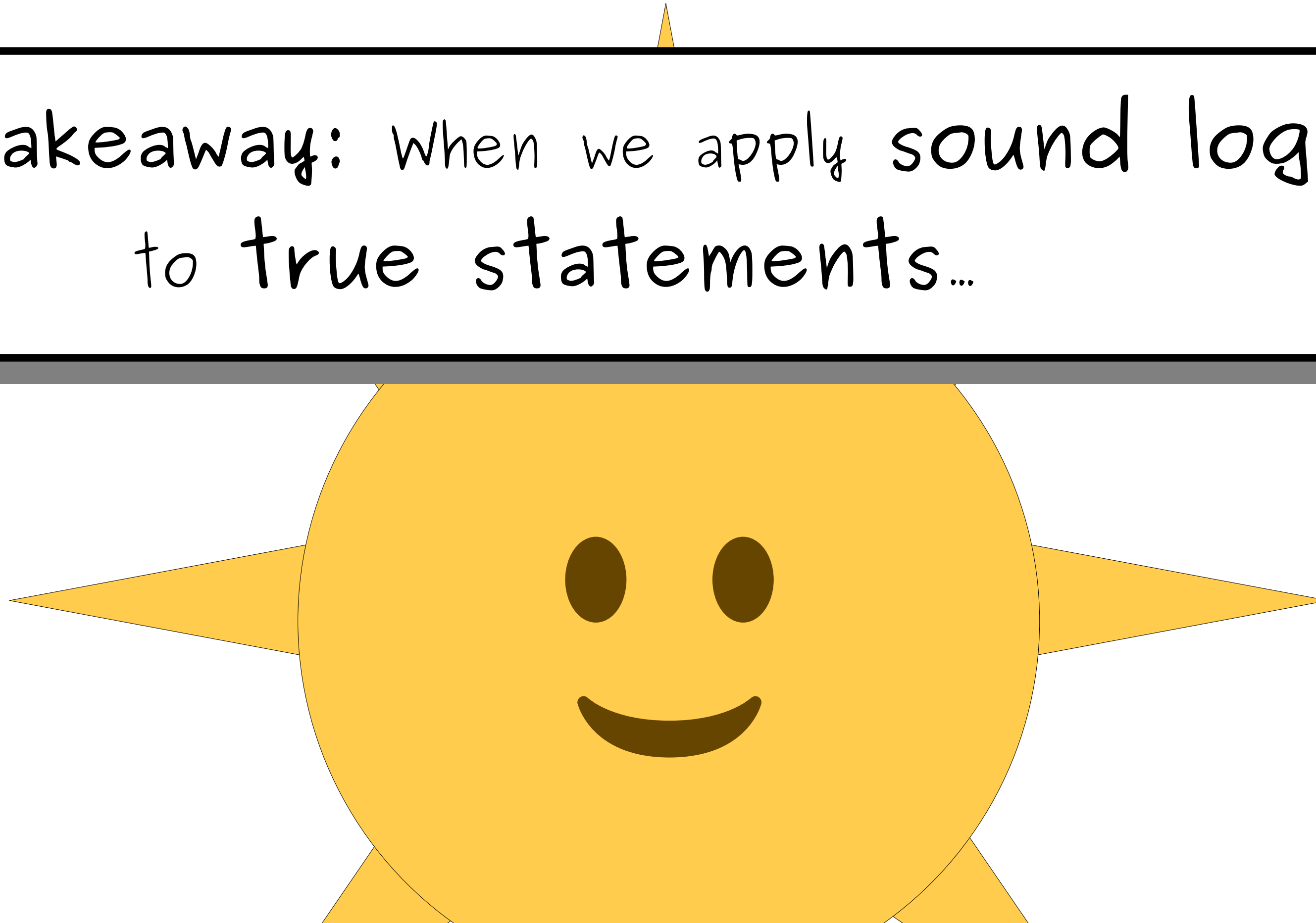
Next, we apply sound logic and rational argument to arrive at other true statements!

Direct Proof



Direct Proof

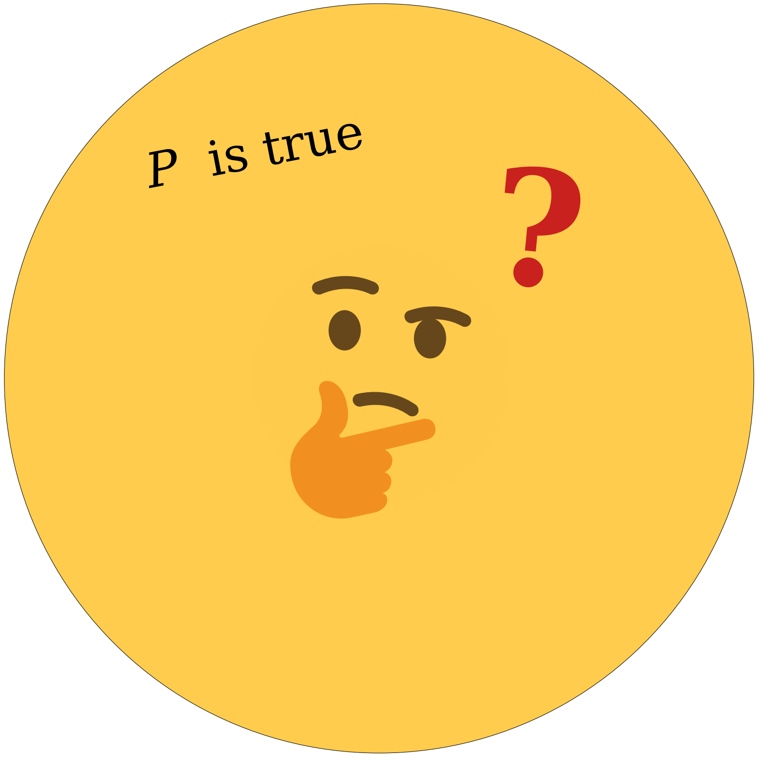
Key Takeaway: When we apply **sound logic** to **true statements**…



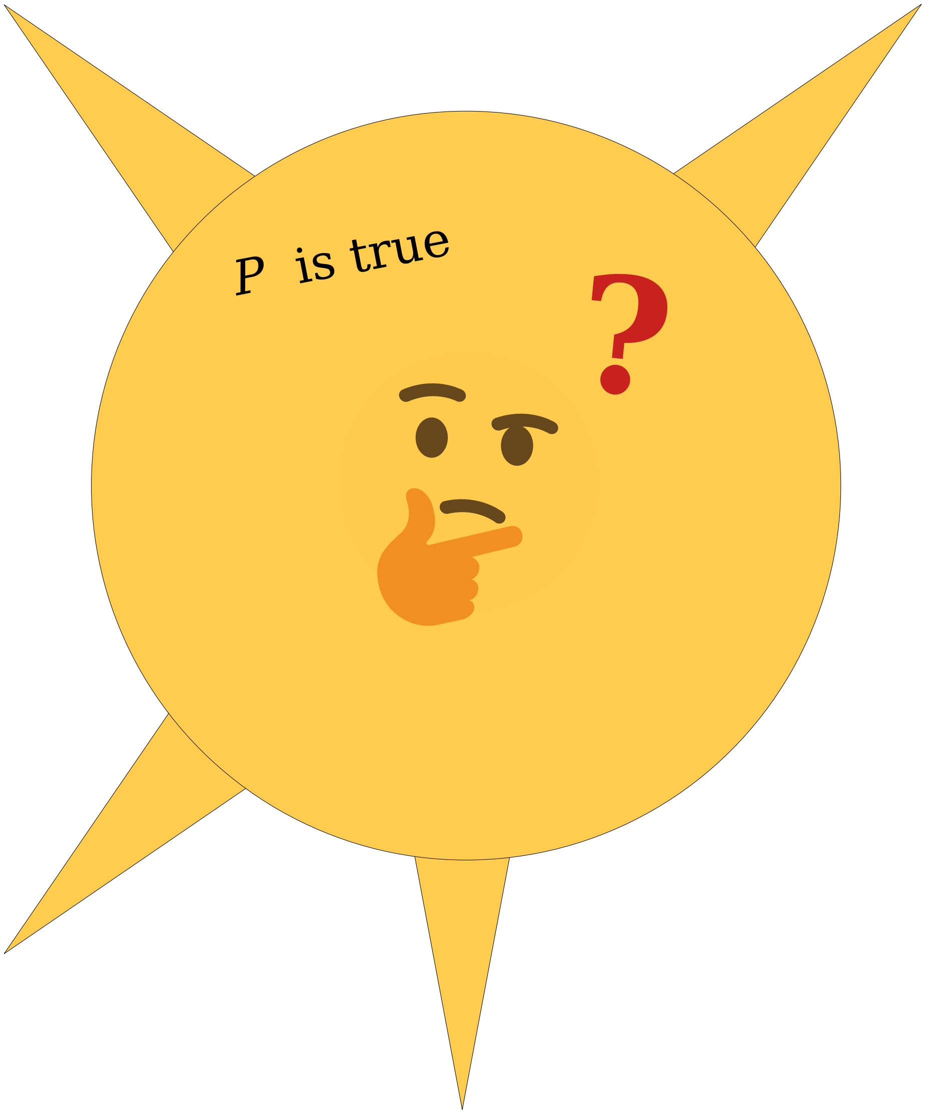
the process **radiates truth** with the power and intensity of a thousand burning suns!

Okay, but…

what if we start with a proposition
whose **truthiness** is **unknown** to us?

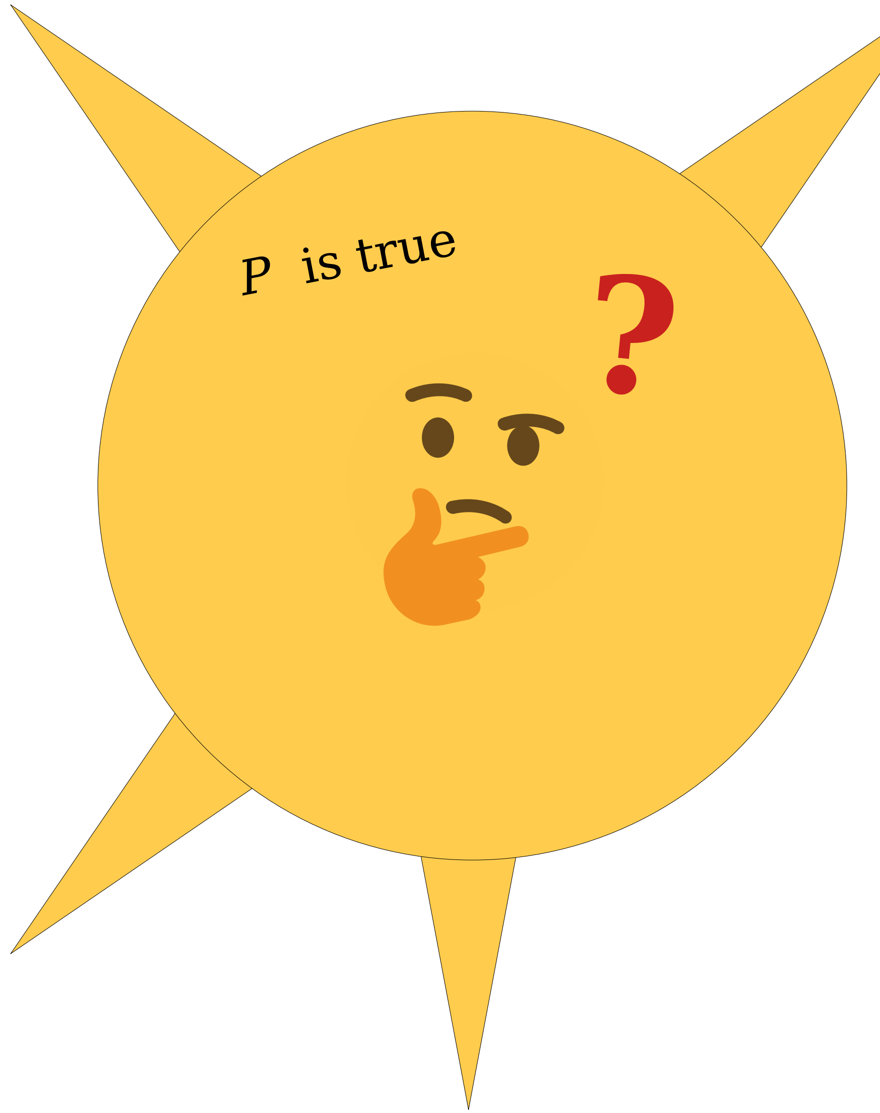

P is true
?

This might radiate **SOME truth.**



This might radiate **SOME truth.**

What does it mean if we start generating statements we **know** are **false**?



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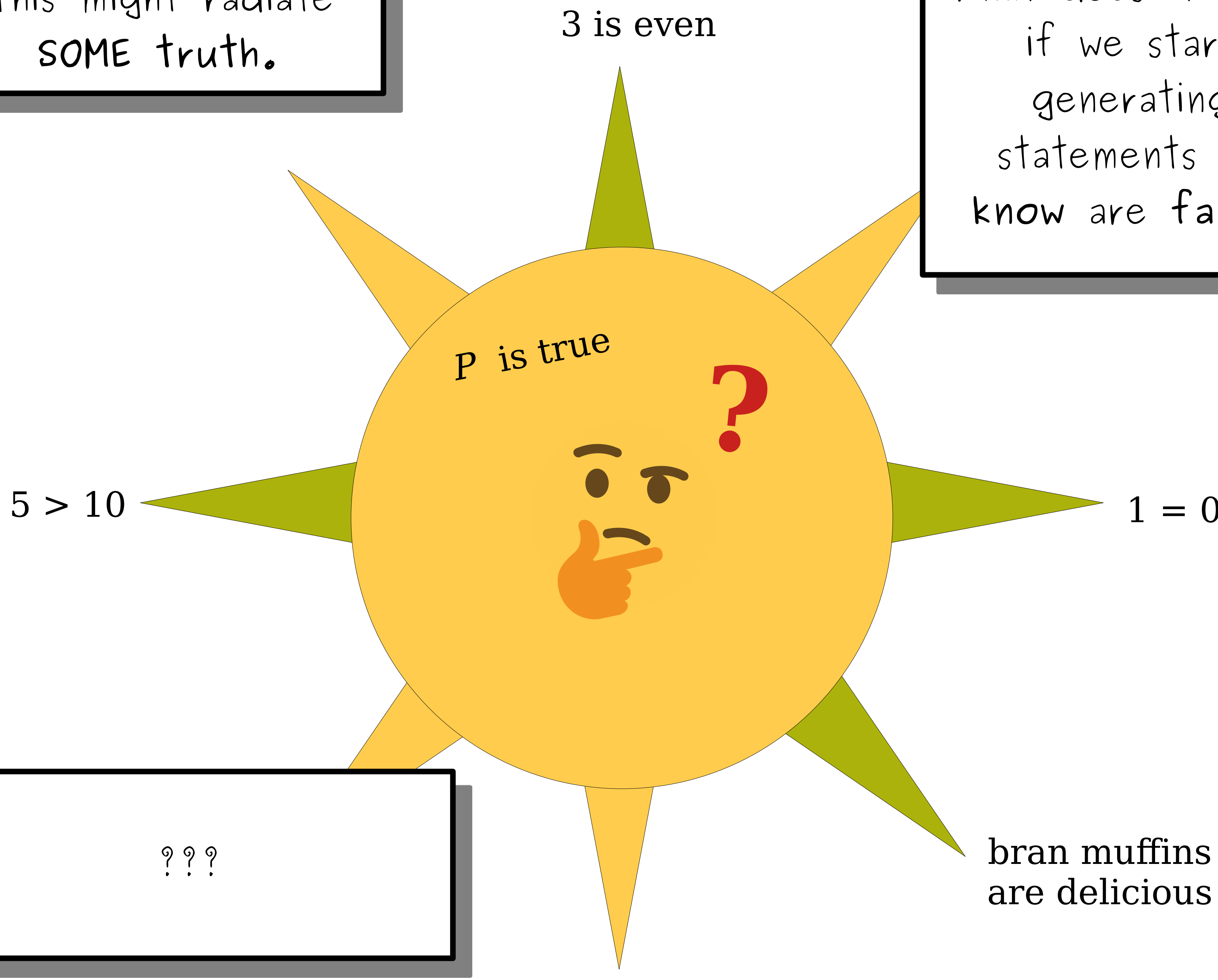

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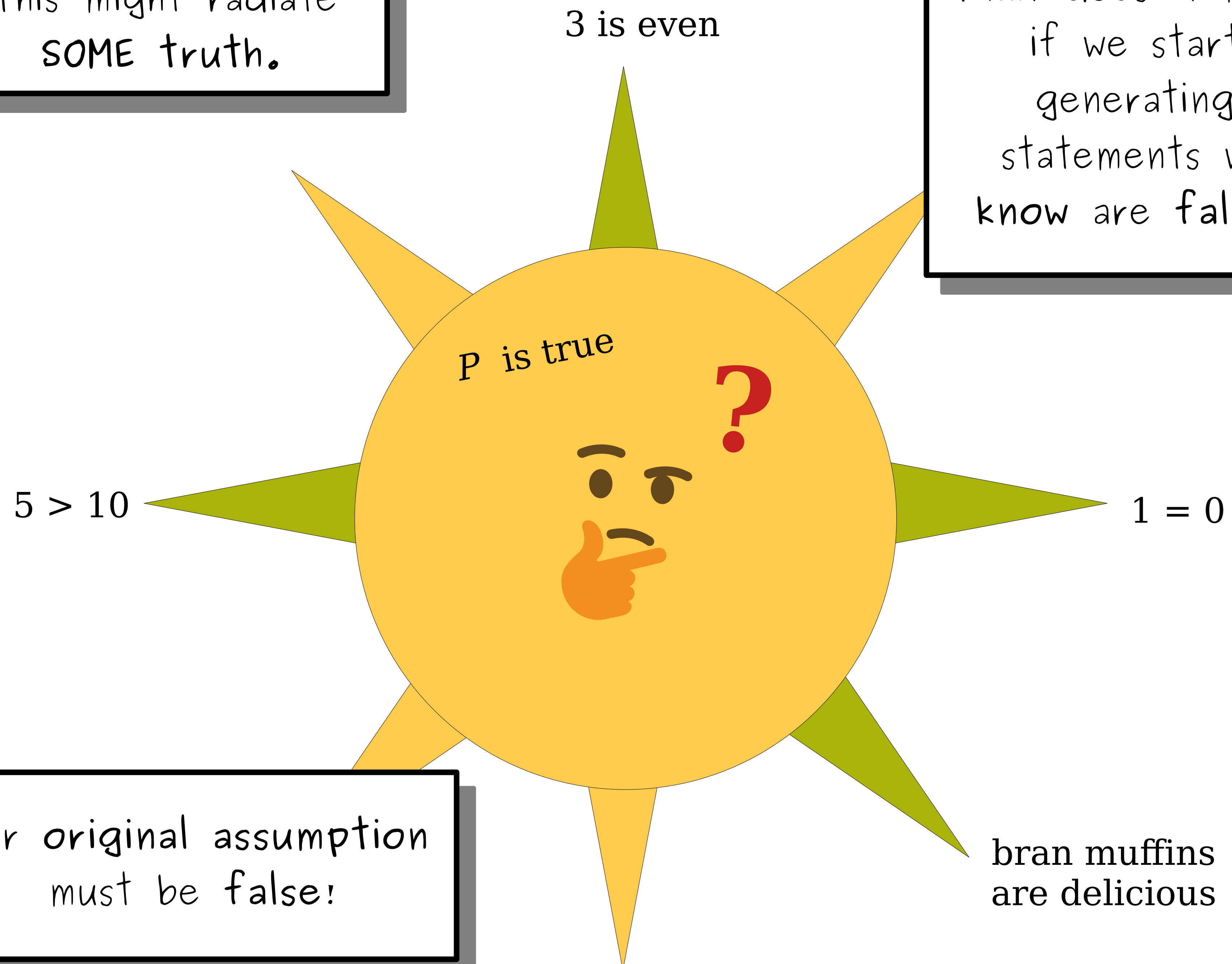
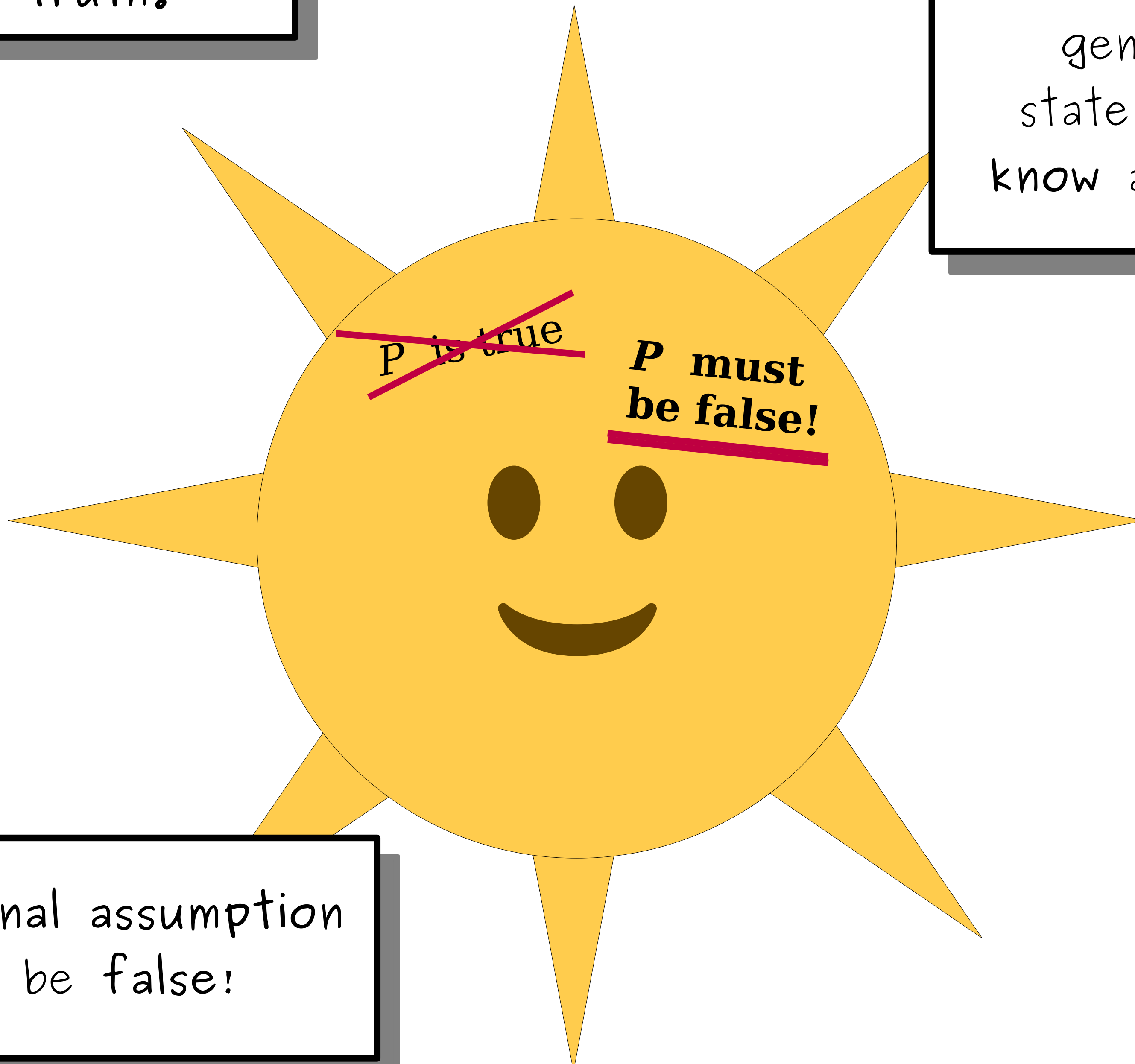


Our **original** **assumption** must be **false**!

This might radiate **SOME truth.**

What does it mean if we start generating statements we **know** are **false**?



Our **original** assumption must be **false!**

This gives rise to a powerful proof technique called **proof by contradiction**!

Suppose we want to use this technique to show that *P* is **true**.

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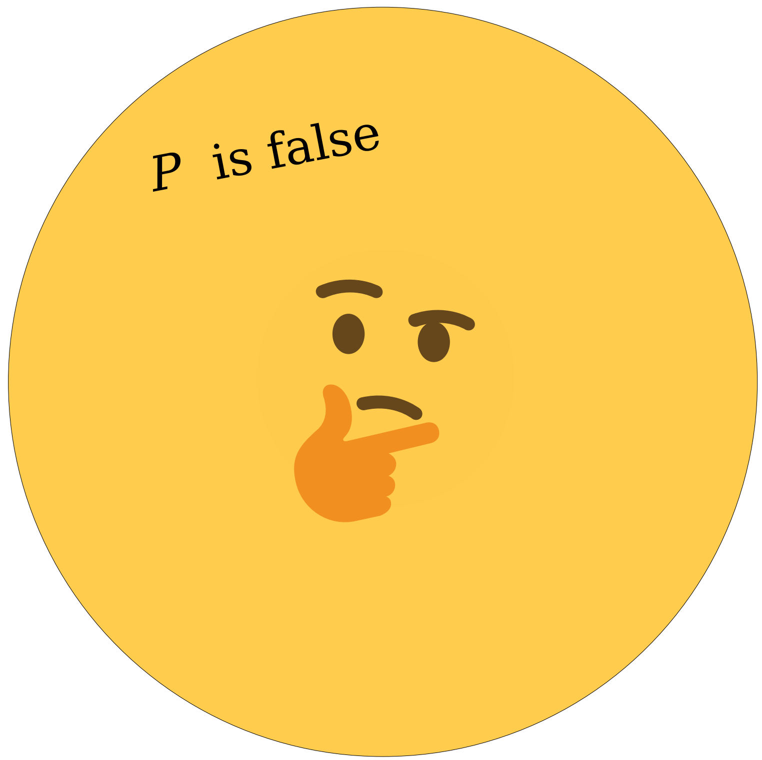
What proposition can we place into the Zone of Uncertainty to accomplish this?

Answer:

The negation of *P*!

i.e., ***P*** is **false**



Suppose we want to use this technique to show that P is **true**.

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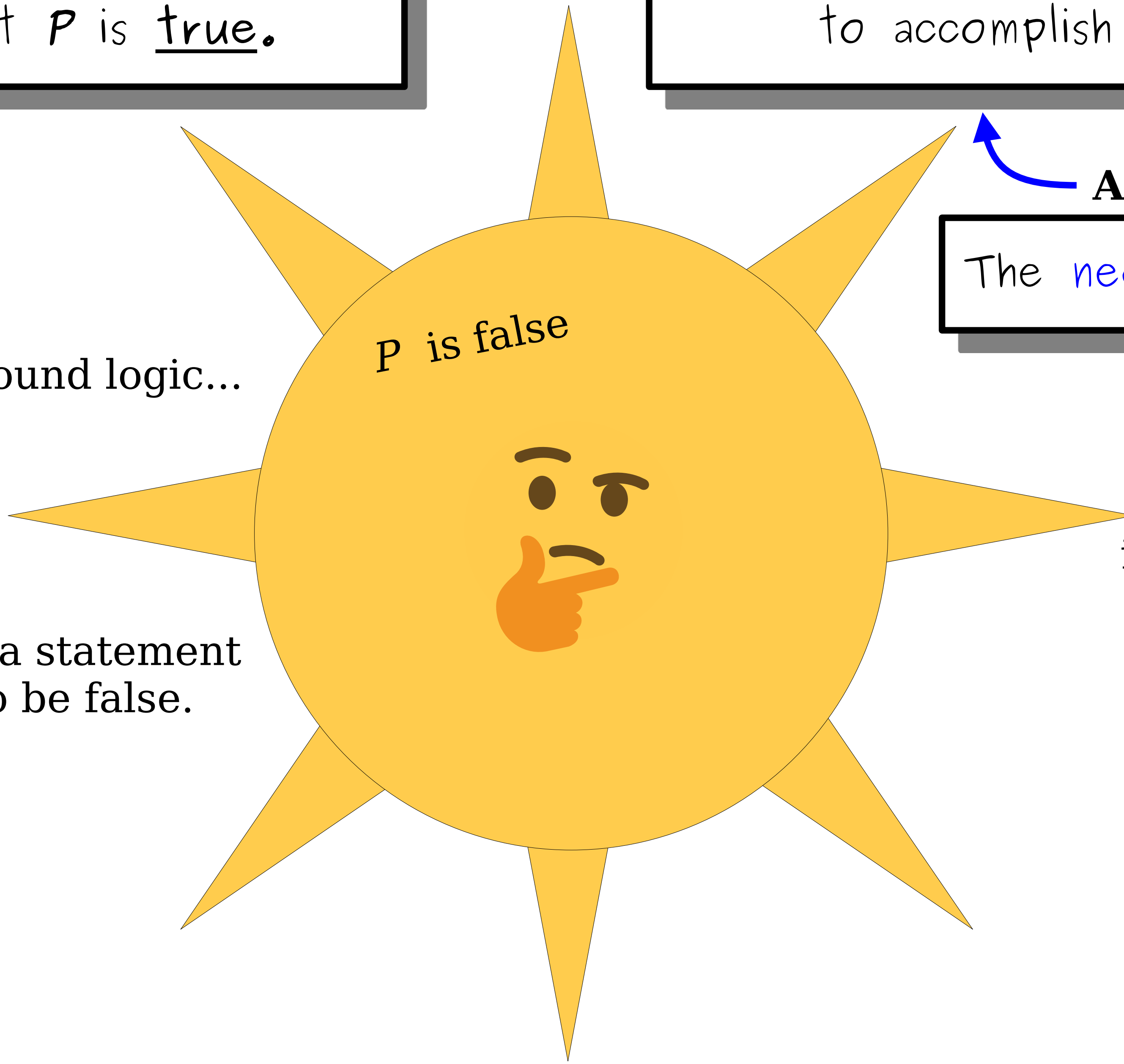
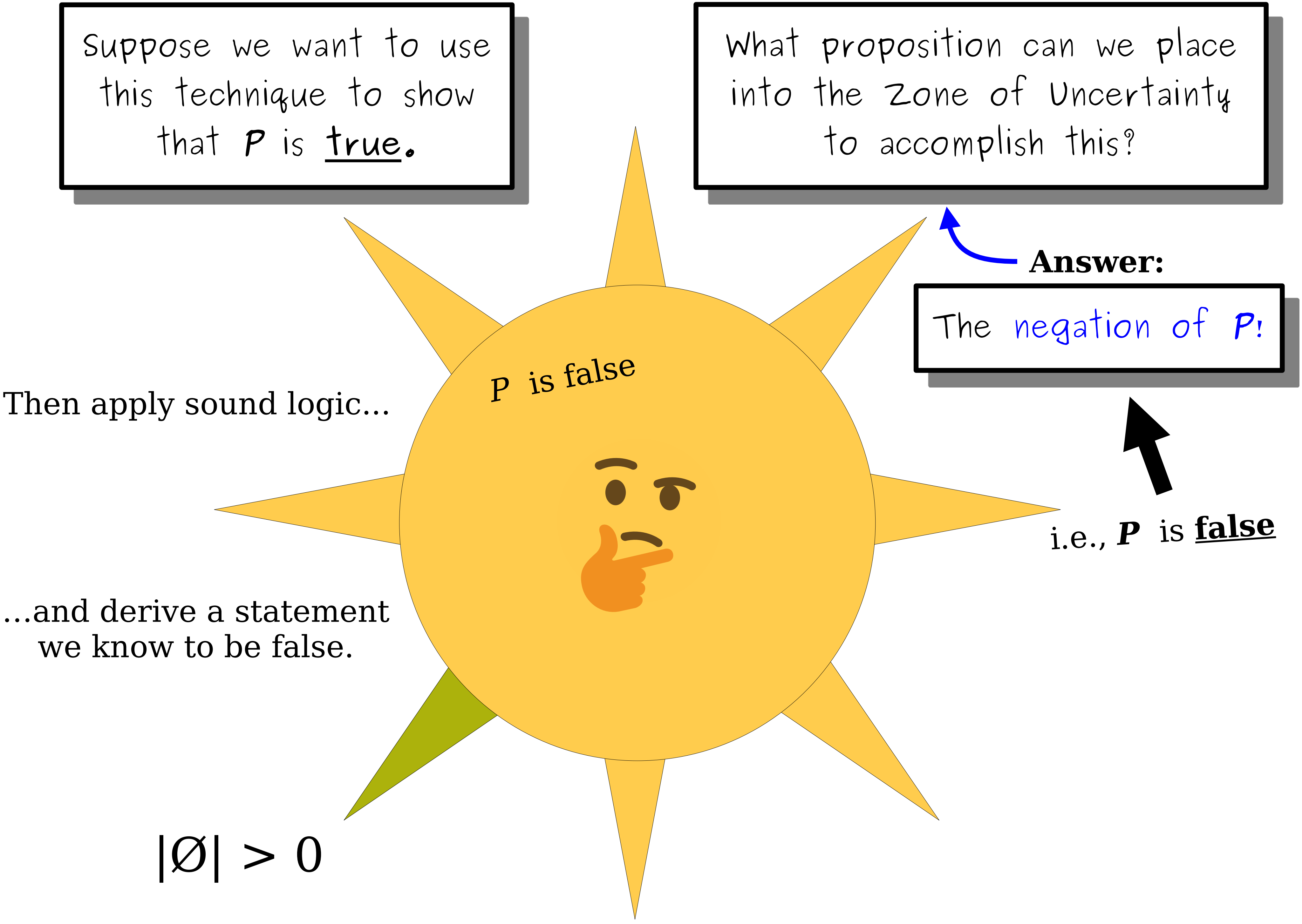
Then apply sound logic...



...and derive a statement we know to be false.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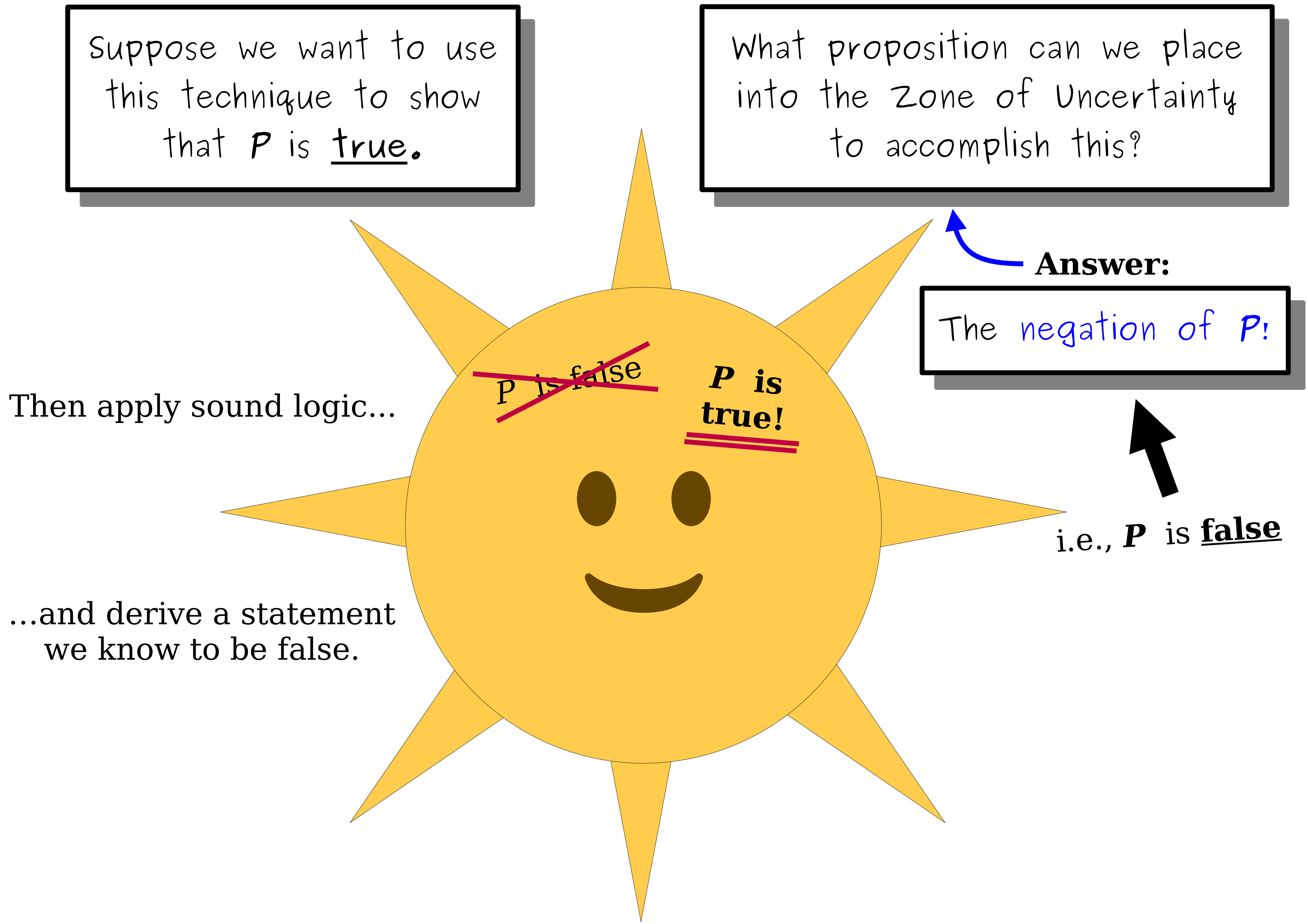

Suppose we want to use this technique to show that P is true.
What proposition can we place into the Zone of Uncertainty to accomplish this?
Answer:
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i.e., P is false
P is false
P is true!
Then apply sound logic...
...and derive a statement we know to be false.

Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
  - $|\wp(\mathbb{N})| > |\mathbb{N}|$
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

Another Example

Latin Squares

- A ***Latin square*** is an $n \times n$ grid filled with the numbers 1, 2, …, n such that every number appears in each row and each column exactly once.
- The ***main diagonal*** of a Latin square runs from the top-left corner to the bottom-right corner.
- A Latin square is ***symmetric*** if the numbers are symmetric across the main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

3	2	5	1	4
2	1	4	5	3
5	4	2	3	1
1	5	3	4	2
4	3	1	2	5

2	5	1	4	3
5	1	3	2	4
1	3	4	5	2
4	2	5	3	1
3	4	2	1	5

Latin Squares

- Notice anything about what's on the main diagonals of these symmetric Latin squares?
- ***Theorem:*** Every odd-sized symmetric Latin square has every number 1, 2, …, *n* on its main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

3	2	5	1	4
2	1	4	5	3
5	4	2	3	1
1	5	3	4	2
4	3	1	2	5

2	5	1	4	3
5	1	3	2	4
1	3	4	5	2
4	2	5	3	1
3	4	2	1	5

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers 1, 2, …, n on its main diagonal.

Proof:

What is the negation of the theorem?

Every symmetric Latin square of odd size n × n has each of the numbers 1, 2, …, n on its main diagonal.

One option:

There is a symmetric Latin square of odd size n × n that does not have one of the numbers 1, 2, …, n on its main diagonal.

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers 1, 2, …, n on its main diagonal.

Proof: Assume for the sake of contradiction that there is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers 1, 2, 3, …, n on its main diagonal.

Notice that we're announcing

1. that this is a proof by contradiction, and
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Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers 1, 2, …, n on its main diagonal.

Proof: Assume for the sake of contradiction that there is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers 1, 2, 3, …, n on its main diagonal. Call the missing number r.

Let k be the number of times r appears above the main diagonal. Since the Latin square is symmetric, there are also k copies of r below the main diagonal. And because r doesn't appear on the main diagonal, that accounts for all copies of r, so there are exactly $2k$ copies of r.

Independently, we know that r appears n times in the Latin square, once for each of its n rows.

Combining these results, we see that $n = 2k$. This means that n is even, contradicting the fact that n is odd. We've reached a contradiction, so our assumption was wrong. Therefore, all symmetric Latin squares of odd size $n \times n$ have each of the numbers 1, 2, …, and n on their main diagonals. ■

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The three key pieces:

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(Intermission)

Time-Out for Announcements!

Problem Set One

- Pset1 goes out today. It's due next Friday at 1 PM.
  - Explore the language of set theory and better intuit how it works.
  - Learn more about the structure of mathematical proofs.
  - Write your first "freehand" proofs based on your experiences.
- As always, ***start early***, and reach out if you have any questions!

Office Hours

- Office hours begin next Monday!
- Feel free to stop on by office hours – no appointment needed.
  - Check out the *Guide to Office Hours* for more information about how our OH system works.
  - The OH calendar is now available on the course website.
- It is ***completely normal*** in this class to need to get help from time to time.
- Feel free to ask clarifying que

PollEV Begins Monday

- Come prepared to log into PollEV. We'll give instructions on Monday, but you'll need a device connected to the wifi.
- We'll use PollEV to record attendance/participation this quarter.
- Our expectation is that those participating in PollEV are physically present in the classroom.

CS103 ACE

- ***CS103 ACE*** is an optional, one-unit companion course to CS103.
- ACE provides additional practice with the course material in a small group setting.
- Meets Tuesdays, 1:30 – 3:20 PM.
- Interested? Apply online using ***this link.*** Deadline is this Friday, 11:59 PM.

Evelyn Yee
(they/them)
ACE Instructor

Working in Pairs

- Starting with Problem Set One, you are allowed to work either individually or in pairs.
    - Each pair should make a **single joint submission**.
- We have advice about how to work effectively in pairs up on the course website; check the “Guide to Partners.”
- Want to work in a pair, but don’t know who to work with? Fill out ***this Google form*** by 5 PM on Friday, and we’ll connect you with a partner sometime on Friday evening.

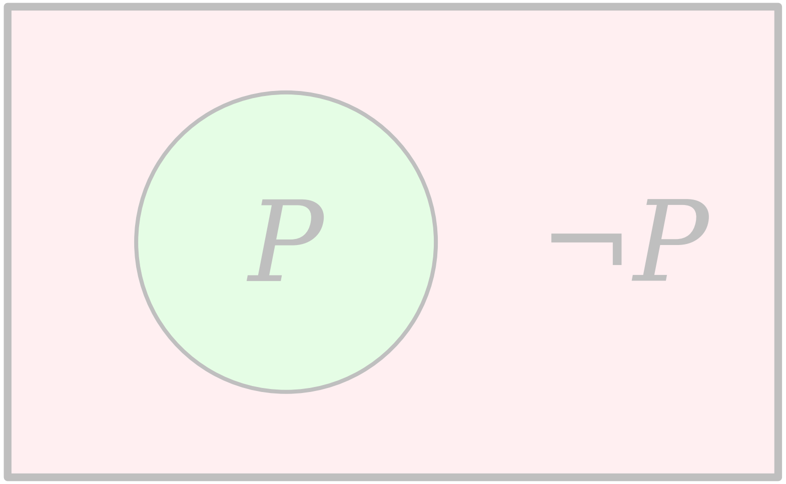
Readings for Today

- First is the ***Proofwriting Checklist***. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.
- Next is the ***Guide to Office Hours***, which talks about how our office hours work and how to make the most effective use of them.
- Finally is the ***Guide to LaTeX***, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.
- Look for the seedling emoji in website menus forn indication of what's new each week.

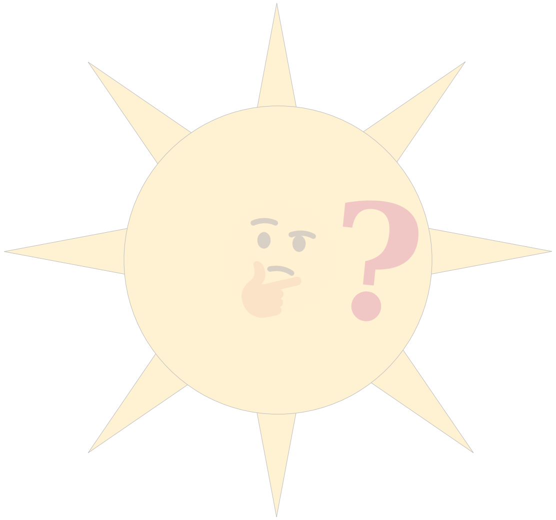
(the lights flash in the atrium)

Back to CS103!

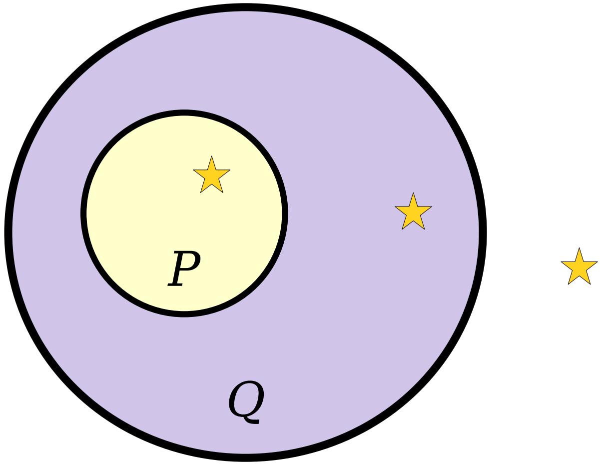
Indirect Proofs: A Story in Four Acts



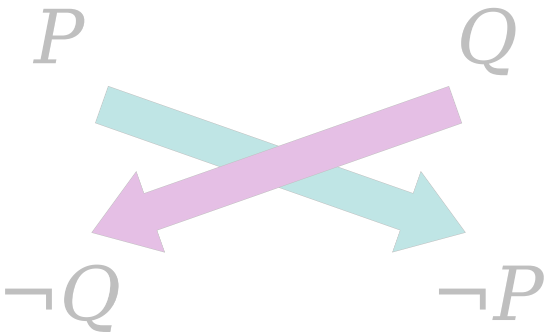
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

Act III

Logical Implication

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the *antecedent*.

This part of the implication is called the *consequent*.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

Let's explore the definition and nature of implication through this example!

"If , then ."

An ***implication*** is a statement of the form
"If P is true, then Q is true."

		What is the status of our "if , then " contract?
✓	✓	contract is not violated ✓
✓	✗	contract **is** violated 😠
✗	✗	contract is not violated ✓
✗	✓	contract is not violated ✓

		What is the status of our “if , then ” contract?
✓	✓	
✓	✗	
✗	✗	contract is not violated ✓
✗	✓	contract is not violated ✓

This one often surprises people! It's part of our **definition** of implication and diverges from how conditional statements work in code.

This one reveals how to negate an implication!

		What is the status of our "if [pig], then [cookie]" contract?
✓	✓	contract is not violated ✓
✓	✗	contract **is** violated 😡
✗	✗	contract is not violated ✓
✗		olated ✓

The only time "if P, then Q" is false is when ***P*** is true and ***Q*** is false.

What Implications Mean

“If there's a rainbow in the sky, then it's raining somewhere.”

- In mathematics, implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
  - Rainbows do not cause rain.

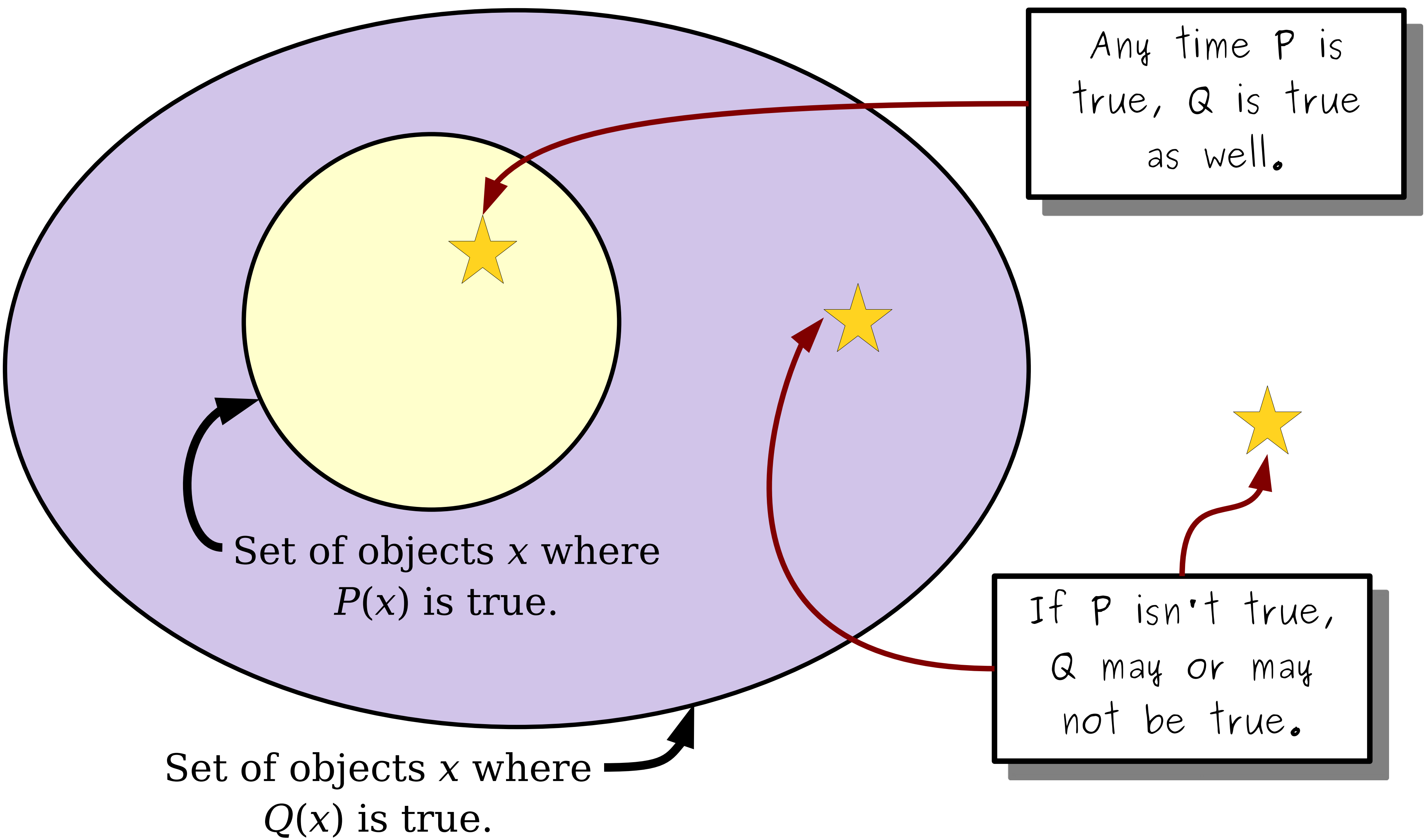
What Implications Mean

- In mathematics, a statement of the form

  For any x, if $P(x)$ is true, then $Q(x)$ is true

  means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



Negating an Implication

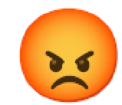
Consider once again the

"if  , then  "

contract.

Question: What has to happen for this contract to be broken?

Answer: A flying pig sings the national anthem, but Sean doesn't throw cookies to the class.

		What is the status of our “if , then ” contract?
✓	✓	contract is not violated ✓
✓	✗	contract **is** violated 
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✗	✓	contract is not violated ✓

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✓	✗	contract **is** violated 😡
✗	✗	contract is not violated ✓
✗	✓	contract is not violated ✓

Negating an Implication

The negation of the statement

"If *P*, then *Q*."

is the statement

"*P* is true, and *Q* is false."

The negation of an implication is not an implication!

Negating an Implication

The negation of the statement

"For any *x*, if *P*(*x*) is true,
then *Q*(*x*) is true"

is the statement

"There is at least one *x* where
***P*(*x*) is true and *Q*(*x*) is false."**

The negation of an implication
is not an implication!

Negating an Implication

The negation of the statement

"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

is the statement

"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."

The negation of an implication is not an implication!

Key take-away!

Negating an Implication

- Example:
  - **If *x* is a puppy, then I love *x*.**
  - **If *x* is a puppy, then I don't love *x*.**
- These statements are **not** negations of one another!
- Consider this scenario:
  - There is some puppy that I love, and there is some puppy that I don't love.
  - Both implications are **false** in this case.
- The negation of the first one is:
  - **There is a puppy that I do not love.**

Negating a universal statement:

For all *x*, *P*(*x*) is true. ⇒ There is an *x* where *P*(*x*) is false.

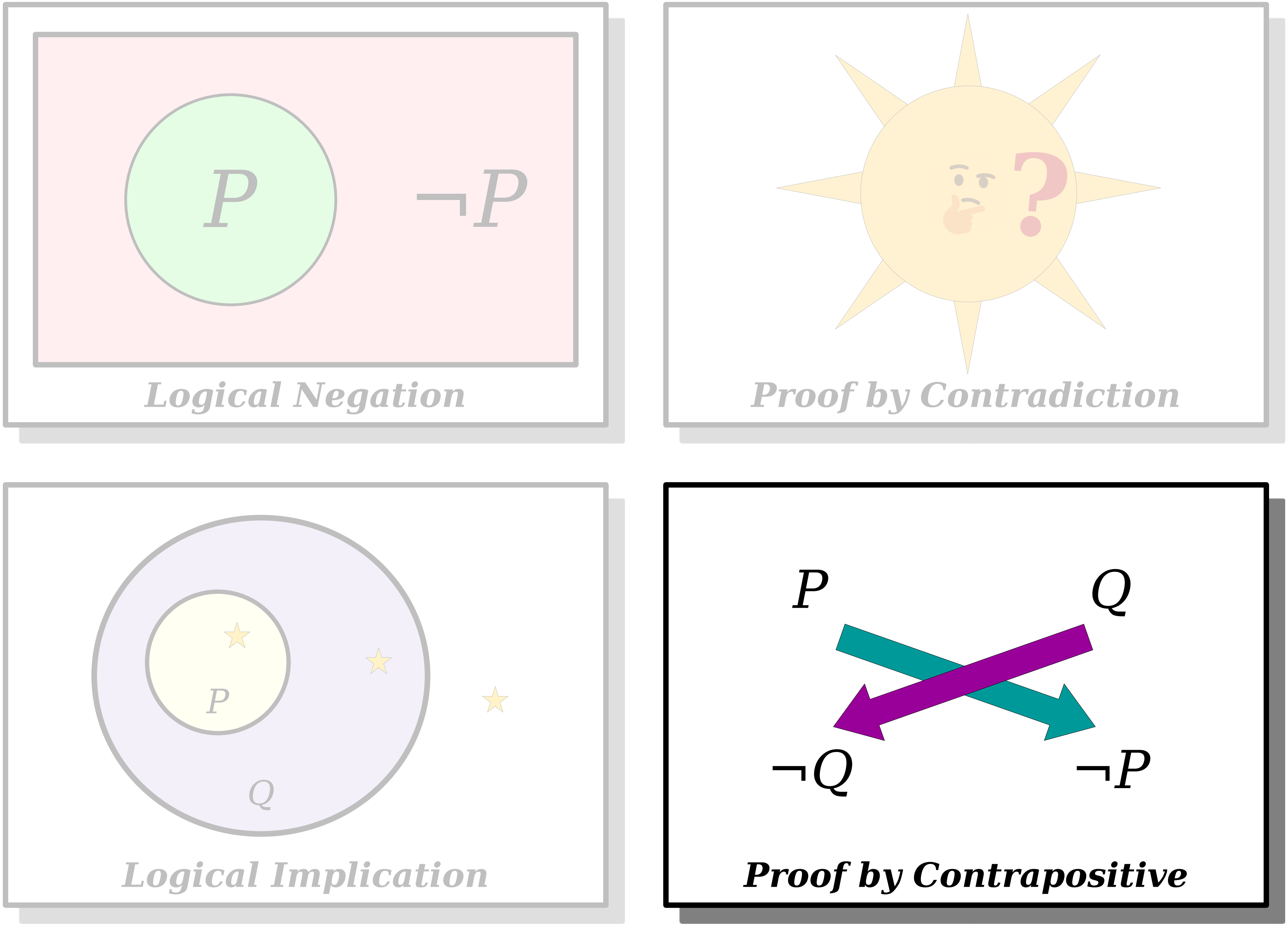
Negating an existential statement:

There exists an *x* where *P*(*x*) is true. ⇒ For all *x*, *P*(*x*) is false.

Negating an implication:

For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true. ⇒ There is an *x* where *P*(*x*) is true and *Q*(*x*) is false.

Indirect Proofs: A Story in Four Acts



Act IV

Proof by Contrapositive

P	*Q*	If *P*, then *Q*.	*Q* is false.	*P* is false.	If *Q* is false, then *P* is false.
✓	✓	✓	✗	✗	✓
✓	✗	✗	✓	✗	✗
✗	✗	✓	✓	✓	✓
✗	✓	✓	✗	✓	✓

P	*Q*	If *P*, then *Q*.	*Q* is false.	*P* is false.	If *Q* is false, then *P* is false.
✓	✓				✓
✓	✗				✗
✗	✗				✓
✗	✓	✓	✗	✓	✓

"If Q is <u>false</u>, then P is <u>false</u>"

is the *contrapositive* of

"If P is <u>true</u>, then Q is <u>true</u>."

To prove the statement

“if *P* is true, then *Q* is true,”

you can choose to instead prove the equivalent statement

“if *Q* is false, then *P* is false,”

if that seems easier.

This is called a ***proof by contrapositive***.

The Contrapositive

- The ***contrapositive*** of the implication

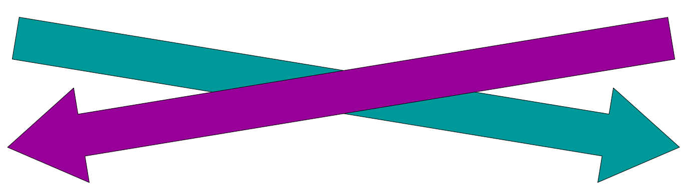
  If *P* is true, then *Q* is true

  is the implication

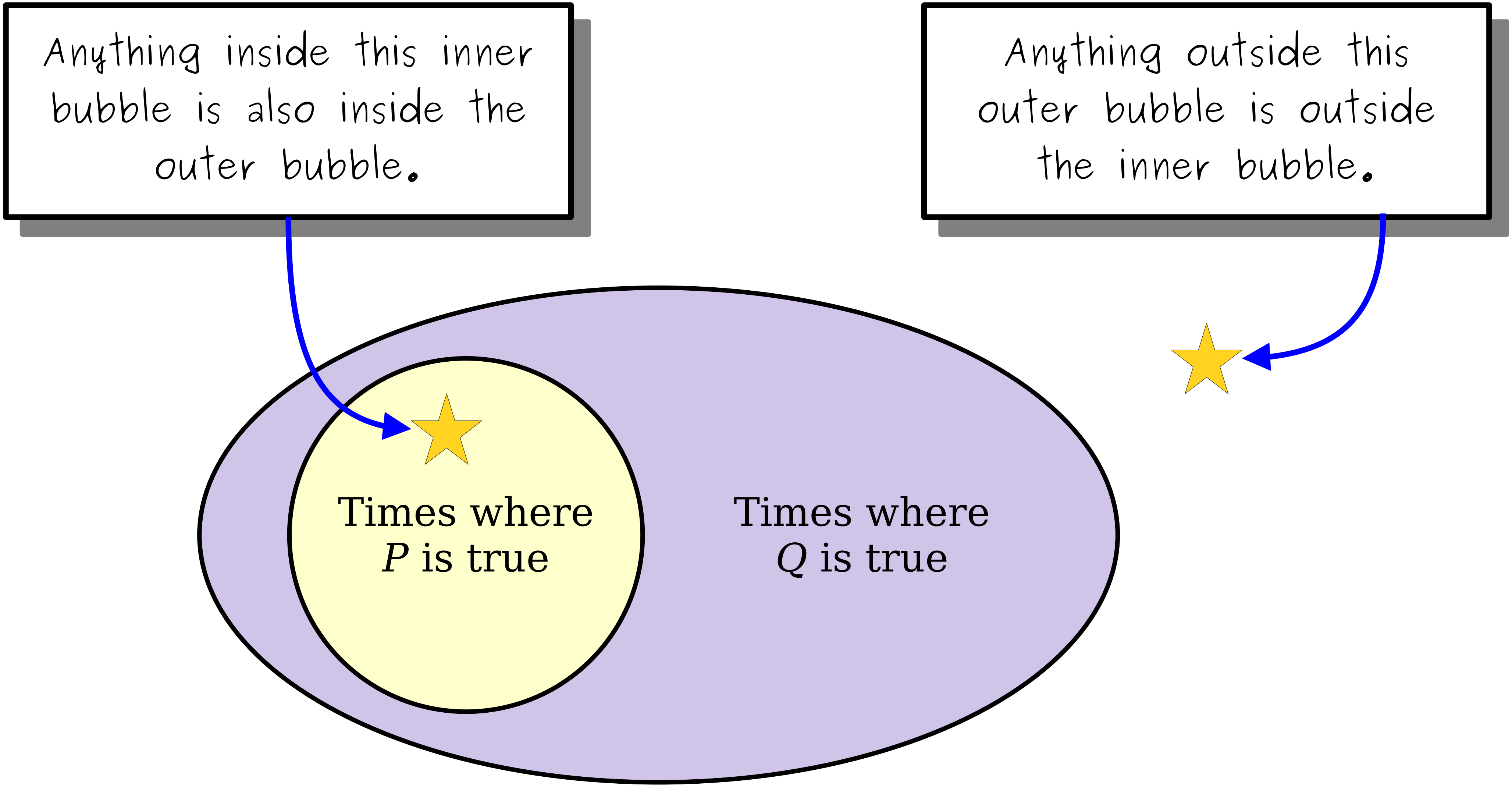
  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.



If P is true, then Q is true.

If Q is false, then P is false.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

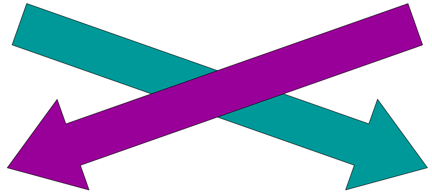
This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.



If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd

We know integer us that

From th (namely means th to show. ■

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

  For any integer n, if n is even, then n^2 is even.

  For any integer n, if n^2 is even, then n is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say

  for any integer n: n is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form

  ***P* if and only if *Q*,**

  you need to prove two separate statements.
  - First, that if *P* is true, then *Q* is true.
  - Second, that if *Q* is true, then *P* is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

What We Learned

- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What's a proof by contradiction?***
  - It's a proof of a statement P that works by showing that P cannot be false.
- ***What's an implication?***
  - It's statement of the form "if P, then Q," and states that if P is true, then Q is true.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if P, then Q" is "if not Q, then not P.")

Your Action Items

- ***Read "Guide to Office Hours," the "Proofwriting Checklist," and the "Guide to LaTeX."***
  - There's a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we'll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what's being asked. That'll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.

Appendix:

Proving Implications by Contradiction

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

    … what does this look like?

Theorem: For any integer n, if n^2 is even, then n is even.

What is the negation of our theorem?

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \tag{2}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

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Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what the negation of the original statement is.
3. Say you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

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$$n = 2k + 1. \qquad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \qquad (2) \end{aligned}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

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Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

    Assume ***P is true*** and ***Q is false***, then derive a contradiction.